Chapter 5 - Vector Spaces

Recall: When we first mentioned $\mathbb{R}^n$

- had some notion of
  1) addition $(\vec{v} + \vec{w})$
  2) c a real #, $\vec{v}$ $\quad c\vec{v}$

These two properties satisfy the following

A1) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ (commutative)

A2) $\vec{v} + (\vec{w} + \vec{z}) = (\vec{v} + \vec{w}) + \vec{z}$ (associative)

A3) There is a $\vec{0}$ with the property that $\vec{v} + \vec{0} = \vec{v}$

A4) For any $\vec{v}$ in $\mathbb{R}^n$ there is some other vector $(-\vec{v})$ such that $v + (-\vec{v}) = \vec{0}$

M1) $a(\vec{v}_1 + \vec{v}_2) = a\vec{v}_1 + a\vec{v}_2$

M2) $(a+b)\vec{v} = a\vec{v} + b\vec{v}$

distributive properties

M3) $a(b\vec{v}) = (ab)\vec{v}$

M4) There is a # (1) such that $1v = \vec{v}$

Def: A vector space V is a set with a defined addition and scalar multiplication, that satisfies

$A_1 - A_4$. and $M_1 - M_4$ above

- we call elements of vectors.

ex) 1) $\mathbb{R}^n$

2) $M_{m \times n}(\mathbb{R})$ = the collection of all $m \times n$ matrices with coefficients in $\mathbb{R}$.

$$0_V = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{pmatrix}$$

3) $\mathbb{R}_n[x]$: the collection of all polynomials up to degree n.

- The 0 vector in $\mathbb{R}_n[x]$ is the # 0!
- Need to define addition:

$$f = a_0 + a_1 x + \cdots + a_n x^n$$
$$g = b_0 + b_1 x + \cdots + b_n x^n$$
$$\overset{\text{define}}{\Longrightarrow} f+g = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$$

- Also need scalar mult:

$$f = a_0 + a_1 x + \cdots + a_n x^n$$
$$cf = ca_0 + ca_1 x + \cdots + ca_n x^n$$

There are a lot of other examples too (these will not be asked of you)

1) $\mathbb{R}[x]$ = the collection of all polynomials.

2) $\mathcal{F}(\mathbb{R})$ = the collection of all functions $f: \mathbb{R} \to \mathbb{R}$

$\cup\!\mid$ $(f+g): \mathbb{R} \to \mathbb{R}$ $\quad (f+g)(x) = f(x) + g(x)$

$0_v : \mathbb{R} \to \mathbb{R}$ such that $0_v(x) = 0$

$\mathcal{C}^0(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \; : \; f \text{ cts} \}$

$\cup\!\mid$

$\mathcal{C}^1(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \; : \; f \text{ is differentiable} \}$

$\cup\!\mid$

$\cup\!\mid$

$\mathcal{C}^\infty(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \; : \; f \text{ is infinitely differentiable} \}$ → smooth

(smooth functions)

Consider the differential equation

$$\frac{d^2f}{dx^2} + f(x) = 0$$

The solution set to this diff eqn is vector space.

Thrm: V be a vector space

1) $0\vec{v} = 0_{\vec{v}}$

2) The $0\vec{v}$ is unique

3) The inverse to any vector $\vec{v}$ is unique.

(think about how to show these)

Section 5.2 - Span / Linear indendence

Def: V be a vector space, let $v_1 \dots v_k$ be vectors in V

1) A linear combo of these vectors is
   an expression $a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_k v_k$
   for $a_1, a_2, \dots, a_k$ in $\mathbb{R}$

2) The set of all possible linear combinations is the span
   denoted $\text{span}(v_1, \dots, v_k)$

3) $W \subseteq V$ is a subspace if it is "closed" under add / scalar mult

4) We say vectors $v_1 \dots v_k$ span a subspace $W \subseteq V$
   if $\text{span}(v_1, \dots, v_k) = W$

Q: How to check if a general vector is in the span of a list of vectors.

ex) $f = 1 + x - 2x^2$, $g = 2 + x - 3x^2$ in $V = \mathbb{R}_2[x]$

Is $2 + 3x - 5x^2$ in $\operatorname{span}(f, g)$?

If so then $2 + 3x - 5x^2 = a_1 f + a_2 g$ for some a_1, a_2

$2 + 3x - 5x^2 = a_1 + a_1 x - 2a_1 x^2 + 2a_2 + a_2 x - 3a_2 x^2$

$2 + 3x - 5x^2 = (a_1 + 2a_2) + (a_1 + a_2)x + (-2a_1 - 3a_2)x^2$

$$\Rightarrow \begin{aligned} a_1 + 2a_2 &= 2 \\ a_1 + a_2 &= 3 \\ -2a_1 - 3a_2 &= -5 \end{aligned} \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 1 & 3 \\ -2 & -3 & -5 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

ex2) $V = M_{2\times 2}(\mathbb{R})$: Consider the vectors

$v = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, $w = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$. Is $\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix}$ in $\operatorname{span}(v, w)$

If so then

$$\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix} = a_1 v + a_2 w \quad \text{for some } a_1, a_2.$$

$$= \begin{pmatrix} 2a_1 & +a_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_2 \\ 2a_1 & 3a_2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 2a_1 & a_1 + a_2 \\ 2a_2 & 3a_2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2a_1 \qquad\quad &= 4 \\ a_1 + a_2 &= 1 \\ 2a_2 &= -2 \\ 3a_2 &= -3 \end{aligned} \longrightarrow \left(\begin{array}{cc|c} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & -3 & -3 \end{array}\right)$$

Linear Independence

Def: A sequence of vectors $v_1, \dots, v_n$ in V are linearly independent (LI) if the only solution to

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0_V$$

is $c_1 = c_2 = \cdots = c_n = 0$

ex) $V = M_{2\times 2}(\mathbb{R})$: Are the vectors

$$m_1 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 1 & 4 \\ 3 & 6 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

LI?

Check: What is solution set to

$$c_1 m_1 + c_2 m_2 + c_3 m_3 + c_4 m_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & 2c_1 \\ 2c_1 & 3c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 & c_2 \\ 3c_2 & 2c_2 \end{pmatrix} + \begin{pmatrix} 3c_3 & 4c_3 \\ 3c_3 & 6c_3 \end{pmatrix} + \begin{pmatrix} 3c_4 & 4c_4 \\ 2c_4 & 3c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 + 2c_2 + 3c_3 + 3c_4 & 2c_1 + c_2 + 4c_3 + 4c_4 \\ 2c_1 + 3c_2 + 3c_3 + 2c_4 & 3c_1 + 2c_2 + 6c_3 + 3c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{aligned} c_1 + 2c_2 + 3c_3 + 3c_4 &= 0 \\ 2c_1 + c_2 + 4c_3 + 4c_4 &= 0 \\ 2c_1 + 3c_2 + 3c_3 + 2c_4 &= 0 \\ 3c_1 + 2c_2 + 6c_3 + 3c_4 &= 0 \end{aligned} \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 3 & 0 \\ 2 & 1 & 4 & 4 & 0 \\ 2 & 3 & 3 & 2 & 0 \\ 3 & 2 & 6 & 3 & 0 \end{array}\right)$$

Finish this! Solve this system to see if they are LI!

ex) $V=\mathbb{R}_2[x]$: Are the vectors $f_1=2+x$, $f_2=3+x^2$, $f_3=x-x^2$ LI?

Check: What is the solution set to

$$c_1f_1+c_2f_2+c_3f_3=0 \leftarrow \text{the \# } 0$$

$$\|$$

$$2c_1+c_1x+3c_2+c_2x^2+c_3x-c_3x^2=0$$

$$\|$$

$$(2c_1+3c_2)+(c_1+c_3)x+(c_2-c_3)x^2=0$$

$$\longrightarrow \begin{aligned} 2c_1+3c_2 \quad &=0 \\ c_1 \quad +c_3 &=0 \\ c_2-c_3 &=0 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

Finish this again! Solve this matrix eqn above to check if they are LI or not!

Why care?

Thm: Suppose that $v_1 \ldots v_k$ are LI.

Then any vector w in $\mathrm{span}(v_1 \ldots v_k)$ has a unique! expression $w = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$

Def: V be vector space. Then a sequence of vectors $v_1 \ldots v_n$ are a basis for V if

1) $\mathrm{span}(v_1, \ldots, v_n) = V$
2) $v_1, \ldots, v_n$ are linearly independent.

ex) 1) $V = M_{3\times 2}(\mathbb{R})$

$$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

2) $\mathbb{R}_3[x]$

$$\mathcal{B} = (1, x, x^2, x^3)$$

3) $V = \mathbb{R}_2[x]$

$$\mathcal{B} = (1, x, x^2)$$

4) V be VS of solutions to diff eqn

$$\frac{d^2f}{dx^2} + f = 0$$

$$\mathcal{B} = \left(\sin(x), \cos(x) \right)$$

5.3 - Dimension

"This requires functional analysis"

$\mathbb{R}^n$		$\mathcal{F}(\mathbb{R})$
$M_{m\times n}(\mathbb{R})$	VS	$C^\infty(\mathbb{R})$
$\mathbb{R}_n[x]$		$\mathbb{R}[x]$

Why are approachable but are not something we'll study in this course?

Thm: 1) Every vector space has a basis!

2) Every basis for a given vector space has the same # of vectors.

Def: V vector space. The <u>\# of vectors</u> in a basis for V is called <u>dimension</u> of V: <u>$\dim V$</u>

ex) $\mathbb{R}^n$: $\dim \mathbb{R}^n = n$

$\mathbb{R}_n[x]$: $\dim \mathbb{R}_n[x] = n+1$

$M_{m\times n}(\mathbb{R})$: $\dim M_{m\times n}(\mathbb{R}) = mn$

ex) $\dim M_{20\times 100}(\mathbb{R}) = 2000$